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Some Results on Ultrametric-Metric-Preserving Functions

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Abstract. Functions whose composition with every ultrametric is a metric are said to be ultrametric-metric-preserving. In this paper, some properties for this set of functions are provided. In addition, several results for this class of functions, that are not valid for functions that preserve the metric, are established. Some examples are given.

Keywords: Ultrametric space, ultrametric-metric-preserving function, metric-preserving function, subbadittive function, uniformly discrete space.

Resumen. Las funciones cuya composición con toda ultramétrica es una metrica, son llamadas funciones que transforman ultramétricas en m étricas. En este trabajo se proporcionan algunas de sus propiedades. Además se proporcionan resultados para estas funciones, que no son validos para funciones que preservan la métrica. Lo anterior se ilustra con algunos ejemplos.

Palabras claves: Espacio ultrametrico, función de preservación- ultramétrica-métrica, función de preservación-métrica, función subaditiva, espacio uniformamente discreto.

1 Introduction and Definitions

Under what conditions on a function $f : [0, \infty) \longrightarrow [0, \infty)$, is true that $f \circ d$ is a metric for every ultrametric space (X, d)? Pongsriiam and Termwut-tipong [1] recently gave theorems concerning ultrametric-metric-preserving functions. The purpose of this paper is to introduce another results about these functions and give some revealing examples.

An *ultrametric space* is a metric space (X, d) satisfying the inequality: for all $x, y, z \in X$, $d(x, y) \leq \max\{d(x, z), d(z, y)\}$. A metric space (X, d) is said to be topologically discrete iff for every $x \in X$ there is an $\varepsilon > 0$ such that $\mathcal{B}_d(x,\varepsilon)=\{x\}$, where $\mathcal{B}_d(x,\varepsilon)$ denote the open ball centered at x and of radius ε. In addition, (X,d) is said to be *uniformly discrete* if there exists an $\varepsilon>0$ such that $\mathcal{B}_d(x,\varepsilon)=\{x\}$ for every $x\in X$.

Now we recall the definitions concerning certain behaviors of functions. Unless noted otherwise, throughout the paper we suppose that $f:[0,\infty)\longrightarrow[0,\infty)$. Then f is said *amenable* if $f^{-1}(0)=\{0\}$, f is *subadditive* if $f(a+b)\leq f(a)+f(b)$ for all $a,b\in[0,\infty)$, f is *convex* if $f((1-t)x+ty)\leq(1-t)f(x)+tf(y)$ for all $x,y\in[0,\infty)$ and $t\in[0,1]$, if $\leq$ is replaced by $<$ and $t\in(0,1)$ then we say that f is *strictly convex.* The function f is called *metric-preserving* if for all metric spaces (X,d), $f\circ d$ is still a metric. This concept has been studied deeply for many authors; see for example [1,2,3,5,6]. We say that f is *ultrametric-metric-preserving* if for all ultrametric spaces (X,d), $f\circ d$ is a metric.

In connection with the metric-preserving functions, the problem arises to investigate the propierties of the ultrametric-metric-preserving functions and compare them with those of metric-preserving functions. We have characterized the continuity of a metric ultrametric-metric-preserving function f at 0, and also we determined the continuity of f at 0 when its domain is a finite product of $[0,\infty)$. In [3,7] it was proved the existence of $f'(0)$ (in the extended sense), where f is a metric-preserving function. Through two examples (5.1 and 5.2), we concluded that the above result is not valid when f is ultrametric-metric-preserving. However, by adding the hypothesis of subadditivity (and continuity in one case) to f, we deduced the existence (in the extended sense) of $f'(0)$.

Recall that a triple (a,b,c) of nonnegative real numbers is called *triangle triplet* if $a\leq b+c$, $b\leq a+c$ and $c\leq a+b$, and it is called *ultra-triangle triplet* if $a\leq\max\{b,c\}$, $b\leq\max\{a,c\}$ and $c\leq\max\{a,b\}$. We denote by $\triangle$, the set of all triangle triplets, and by $\triangle_\infty$, the set of all ultra-triangle triplets.

Now we are ready to state the results we will use in the proof of our theorems.

THEOREM 1.1. $[3,6]$ *Let f be amenable. Then the following statements are equivalent:*

(*i*) *f is metric-preserving*;

(*ii*) *If* $(a,b,c)\in\triangle$ *then* $(f(a),f(b),f(c))\in\triangle$. □

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PROPOSITION 1.2. [1] *If f is an ultrametric-metric-preserving function then f is amenable.* $\square$

LEMMA 1.3. [1] *If $(a, b, c) \in \triangle_\infty$, then $a \leq b = c$ or $b \leq c = a$ or $c \leq a = b$.* $\square$

LEMMA 1.4. [1] *If (X, d) is an ultrametric space and $x, y, z \in X$ then the triple $(d(x, y), d(x, z), d(z, y))$ is an ultra-triangle triplet. Conversely, if (a, b, c) is an ultra-triangle triplet, then there exist an ultrametric space (X, d) and $x, y, z \in X$ such that $a = d(x, y)$, $b = d(x, z)$, and $c = d(z, y)$.* $\square$

THEOREM 1.5. [1] *Let f be amenable. Then the following statements are equivalent:*

(i) *f is ultrametric-metric-preserving;*

(ii) *If $(a, b, c) \in \triangle_\infty$ then $(f(a), f(b), f(c)) \in \triangle$;*

(iii) *for each $0 \leq a \leq b$, $f(a) \leq 2f(b)$.* $\square$

Let T be a nonempty set of indices. We say that $f : [0, \infty)^T \longrightarrow [0, \infty)$ is amenable if $f^{-1}(0) = \{0\}$, and it is *ultrametric-metric-preserving* if for each indexed family $\{(X_t, d_t)\}_{t\in T}$ of ultrametric spaces the function $f \circ d$ is a metric on the set $\prod_{t\in T} X_t$, where $d : \left(\prod_{t\in T} X_t\right)^2 \longrightarrow [0, \infty)^T$ is given by $d((x_t)_{t\in T}, (y_t)_{t\in T}) = (d_t(x_t, y_t))_{t\in T}$ which is denoted by $d = (d_t)_{t\in T}$. For $x, y \in [0, \infty)^T$ we will say that

$$x < y \quad (\text{resp. } x \leq y) \quad \text{if and only if} \quad x_t < y_t \quad (\text{resp. } x_t \leq y_t) \quad \text{for all } t \in T,$$

PROPOSITION 1.6. *If $f : [0, \infty)^T \longrightarrow [0, \infty)$ is an ultrametric-metric-preserving function then f is amenable.*

PROOF. Let $x = (x_t)_{t\in T} \in [0, \infty)^T$. For each $t \in T$ we put $X_t = \{u_t, v_t, w_t\} \subset \mathbb{R}^2$, being $u_t = (-\frac{x_t}{2}, 0)$, $v_t = (\frac{x_t}{2}, 0)$ and $w_t = (0, x_t)$. Let $d_t = d_\infty \mid_{X_t}$ be the restriction on X_t of the metric $d_\infty((a_1, b_1), (a_2, b_2)) = \max\{\mid a_1 - a_2 \mid, \mid b_1 - b_2 \mid\}$. Then $d_t(u_t, v_t) = d_t(u_t, w_t) = d_t(v_t, w_t) = x_t$. Therefore (X_t, d_t) is an ultrametric space. By hypothesis, $f \circ d$ is a metric on $\left(\prod_{t\in T} X_t\right)^2$ where $d =$

$(d_t)_{t\in T}$. In consecuence, for $u, v \in [0,\infty)^T$ we have $f(0) = f(\rho(u,u)) = (f \circ \rho)(u,u) = 0$ and $(f\circ\rho)(u,v) = f(\rho(u,v)) = f((d_t(u_t,v_t)_{t\in T}) = f((x_t)_{t\in T}) = f(x) = 0$ which implies $u = v$, that is $x = 0$. This shows that f is amenable as desired.

□

THEOREM 1.7. *Let $f : [0,\infty)^T \longrightarrow [0,\infty)$ be amenable. Then the following statements are equivalents*

(*i*) *f is ultrametric-metric-preserving*;

(*ii*) *If $(a_t, b_t, c_t) \in \triangle_\infty$ for each $t \in T$ then*

$$(f((a_t)_{t\in T}), f((b_t)_{t\in T}), f((c_t)_{t\in T})) \in \triangle;$$

(*iii*) *for each $0 \le a \le b$, $f(a) \le 2f(b)$.*

PROOF. $(i) \implies (ii)$. Let $t \in T$ and $(a_t, b_t, c_t) \in \triangle_\infty$. By Lemma 1.4, there exist an ultrametric space (X_t, d_t) and $x_t, y_t, z_t \in X_t$ such that $a_t = d_t(x_t, y_t), b_t = d_t(x_t, z_t), c_t = d_t(z_t, y_t)$. Let $d = (d_t)_{t\in T}$. So $f\circ d$ is a metric on $\prod_{t\in T} X_t$ by (i). Therefore we conclude
$(f\circ d(x,y), f\circ d(x,z), f\circ d(z,y)) = (f((a_t)_{t\in T}), f((b_t)_{t\in T}), f((c_t)_{t\in T})) \in \triangle$, as required.
$(ii) \implies (iii)$. Let $0 \le a \le b$. Then $(a_t, b_t, b_t) \in \triangle_\infty$ for each $t \in T$. By hypothesis $(f((a_t)_{t\in T}), f((b_t)_{t\in T}), f((b_t)_{t\in T})) \in \triangle$. Hence $f(a) \le 2f(b)$, as desired.
$(iii) \implies (i)$. Let $t \in T$. Choose $a_t, b_t \in [0,\infty)$, with $a_t \le b_t$. Since $(a_t, b_t, b_t) \in \triangle_\infty$, by Lemma 1.4 there exist an ultrametric space (X_t, d_t) and $u_t, v_t, w_t \in X_t$ such that $a_t = d_t(u_t, v_t), b_t = d_t(u_t, w_t) = d_t(w_t, v_t)$. We will prove that $f\circ d$ is a metric on $\prod_{t\in T} X_t$, where $d(x,y) = (d_t(x_t,y_t))_{t\in T}$. Since f is amenable, $f\circ d(x,y) = 0$ if and only if $x = y$. So it remains to show that the triangle inequality holds for $f\circ d$. By Lemma 1.4, we have $(d_t(x_t,y_t), d_t(x_t,z_t), d_t(z_t,y_t)) \in \triangle_\infty$. From Lemma 1.3, we can assume without loss of generality that $d_t(x_t,y_t) \le d_t(y_t,z_t) = d_t(z_t,x_t)$. By hypothesis, $f\circ d(x,y) = f((d_t(x_t,y_t)_{t\in T})) \le 2f(d_t(y_t,z_t)_{t\in T}) = f(d_t(y_t,z_t)_{t\in T}) + f(d_t(z_t,x_t)_{t\in T}) = f\circ d(y,z) + f\circ d(z,y)$. Hence (i) holds. This completes the proof.

□

2 Some Basic Results

We will denote for $\mathcal{UM}$ the set of all ultrametric-metric-preserving functions, by $\mathcal{M}$ the set of the metric-preserving functions, and by $\mathcal{S}$ the set of the subadditive functions. Since $\triangle_\infty \subset \triangle$, it is clear the contention $\mathcal{M} \subset \mathcal{UM}$. Also, it is known that $\mathcal{M} \subset \mathcal{S}$. The next example shows that $\mathcal{S} \nsubseteq \mathcal{UM}$ and $\mathcal{UM} \nsubseteq \mathcal{S}$.

EXAMPLE 2.1. *Let $f, g : [0, \infty) \longrightarrow [0, \infty)$ be given by*

$$f(x) = x^2, \qquad g(x) = \frac{x}{1+x^2}. \tag{1}$$

The function f is increasing. So, by Theorem 1.5, $f \in \mathcal{UM}$. Also, we have that $f(1+2) > f(1) + f(2)$, that is, $f \notin \mathcal{S}$.
It is easy see that $g(a+b) \leq g(a) + g(b)$ for all $a, b \in [0, \infty)$, and therefore $g \in \mathcal{S}$. By Proposition 2.3 below $g \notin \mathcal{UM}$.

$\square$

The next example shows a continuous function in the set $(\mathcal{UM} \cap \mathcal{S} \setminus \mathcal{M})$.

EXAMPLE 2.2. Let $f(x) = \phi(x - 2p) + p$ if $2p \leq x \leq 2(p+1)$ for $p = 0, 1, 2, \ldots$, where ϕ is given by

$$\phi(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1; \\ 3 - x, & \text{if } 1 \leq x \leq 2. \end{cases} \tag{2}$$

We will show that $f \in \mathcal{UM}$ by applying Theorem 1.5. So we let $0 \leq a \leq b$. The posibility $a = 0$ is obvious, so consider $a > 0$. There exists $k \in \mathbb{N}$ such that $a \in I_k = (k-1, k]$.

(i) *k is even.* If $b \geq k + \frac{1}{2}$ then $f(a) \leq f(b)$. So, we can suppose $k - 1 < a \leq b < k + \frac{1}{2}$. Since $f([k-1, k+\frac{1}{2}]) = [\frac{k}{2}, \frac{k}{2}+1]$, we have $\mid f(a) - f(b) \mid \leq 1$. So, $f(a) \leq f(b) + 1 \leq f(b) + \frac{k}{2} \leq 2f(b)$.

(ii) *k is odd.* If $a, b \in I_k$ then $f(a) \leq f(b)$. Now, we assume that $b \notin I_k$. If $b \geq k + \frac{3}{2}$ then $f(a) \leq f(b)$. Let $k < b < k + \frac{3}{2}$. We only consider the posibility non trivial $k - \frac{1}{2} \leq a$. By definition of f we deduce $f([k - \frac{1}{2}, k + \frac{3}{2})) = [\frac{k+1}{2}, \frac{k+3}{2})$. Therefore $\mid f(a) - f(b) \mid \leq 1$. Thus $f(a) \leq 1 + f(b) \leq \frac{k+1}{2} + f(b) \leq 2f(b)$.

In any case, $f(a) \leq 2f(b)$ holds. Hence $f \in \mathcal{UM}$. To see that $f \notin \mathcal{M}$ it is sufficient to observe that $(2,2,3) \in \triangle$ but $(f(2), f(2), f(3)) \notin \triangle$.
Next,we shall prove that f is subadditive. Let $0 < a \leq b$. There exist integers $m \leq n \leq k$ so that $a \in I_m$, $b \in I_n$ and $a+b \in I_k$. We have the following cases

(1) $m + n = k$. Assume k is odd. Then m is even and n is odd or vice versa. Namely m is even and n is odd. Hence

$$\begin{aligned} f(a+b) &= 2(a+b) - \frac{3}{2}(m+n-1) \\ &= f(a) + f(b) + 3(a-m) \\ &\leq f(a) + f(b). \end{aligned}$$

For the posibility that m is odd and n is even, we can also get easily that $f(a+b) = f(a) + f(b) + 3(b-n) < f(a) + f(b)$.
Now, suppose that k is even. If m and n are odd then

$$\begin{aligned} f(a+b) &= \frac{3}{2}k - (a+b) \\ &= f(a) + f(b) + 3((m+n) - (a+b) - 1) \\ &= f(a) + f(b) + 3((k-1) - (a+b)) \\ &< f(a) + f(b). \end{aligned}$$

It is easy to see that $f(a+b) = f(a) + f(b)$ provided m and n are even.

(2) $m+n-1 = k$. When k is odd , we can obtain similarly to the previous case

$$f(a+b) = \begin{cases} 3((a+b)-k) + f(a) + f(b) \leq f(a) + f(b), & \text{if } n, m \text{ even;} \\ f(a) + f(b), & \text{if } n, m \text{ odd.} \end{cases}$$

In addition, if k is even then

$$f(a+b) = \begin{cases} 3((n-1)-b) + f(a) + f(b) < f(a) + f(b), & \text{if } m \text{ even, } n \text{ odd;} \\ 3((m-1)-a) + f(a) + f(b) < f(a) + f(b), & \text{if } m \text{ odd, } n \text{ even.} \end{cases}$$

This completes the proof. Therefore f is subadditive.

□

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The conclusion of the following proposition is valid when f is metric-preserving. Its proof can be found in [3]. However, the result is also true in the case that f is ultrametric-metric-preserving.

PROPOSITION 2.3. *Let $f : [0, \infty) \longrightarrow [0, \infty)$ be an ultrametric-metric-preserving function. Then for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $f^{-1}[0, \delta) \subset [0, \varepsilon)$.*

PROOF. If the assertion is false, then there are $\varepsilon > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n > \varepsilon$ for all n and $\{f(x_n)\}_{n\in\mathbb{N}} \longrightarrow 0$. Let n_0 be such that $f(x_{n_0}) < \dfrac{f(\varepsilon)}{2}$. Then $\varepsilon < x_{n_0}$ and $f(\varepsilon) > 2f(x_{n_0})$, which contradicts to Theorem 1.5.

□

PROPOSITION 2.4. *Let $f : [0, \infty) \to [0, \infty)$ amenable. If there exists an nondecreasing function $g : [0, \infty) \to [0, \infty)$ such that $g(x) \leq f(x) \leq 2g(x)$ for all $x \in [0, \infty)$, then f is ultrametric-metric-preserving.*

PROOF. Let $0 \leq a \leq b$. Then

$$f(a) \leq 2g(a) \leq 2g(b) \leq 2f(b).$$

So, the result follows from Theorem 1.5.

□

By Proposition 2.4 $f(x) = x + x \mid \sin x \mid$ for $x \geq 0$ is ultrametric-metric-preserving because $f(x) \in [x, 2x]$. In this case, we also have $f \notin \mathcal{M}$. In fact, $(\frac{\pi}{2}, \pi, \frac{3}{2}\pi)$ is a triangle triplet, but $(f(\frac{\pi}{2}), f(\pi), f(\frac{3}{2}\pi))$ is not.

3 Continuity

In this section, we will investigate aspects of the continuity of the functions in $\mathcal{UM}$. The continuity of functions in $\mathcal{M}$ has already been studied [1,3,5].

THEOREM 3.1. [3] *Suppose $f : [0, \infty) \longrightarrow [0, \infty)$ is metric-preserving and continuous at 0. Then f is continuous on $[0, \infty)$.*

□

What conditions must satisfy a function $f \in \mathcal{UM} \setminus \mathcal{M}$ which is continuous at 0, in order to be continuous on $[0, \infty)$? In the example 3.8 below, we give a function in $(\mathcal{UM} \cap \mathcal{S}) \setminus \mathcal{M}$, continuous at 0, and discontinuous at the points $\frac{n}{2}$ for $n \geq 2$.

THEOREM 3.2. [1] *Let $f : [0, \infty) \longrightarrow [0, \infty)$ be ultrametric-metric-preserving. Then f is continuous at 0, if and only if for every $\varepsilon > 0$ there exists an $x > 0$ such that $f(x) < \varepsilon$.*

□

The next result is a caracterization of the discontinuity at 0 of an ultrame tric-metric-preserving function.

THEOREM 3.3. *Let $f : [0, \infty) \longrightarrow [0, \infty)$ be an ultrametric-metric-preserving function. Then f is discontinuous at 0 if and only if $f \circ d$ is an uniformly discrete metric for every ultrametric d.*

PROOF. Let (X, d) be an ultrametric space. Suppose that f is discontinuous at 0. By Theorem 3.2 there exists an $\varepsilon > 0$ such that $f(z) > \varepsilon$ for all $z > 0$. Then $\mathcal{B}_{f \circ d}(x, \varepsilon) = \{x\}$ for each $x \in X$ as required. Inversely, we assume that $f \circ d$ is an uniformly discrete metric for every ultrametric d. Let $x > 0$. Put $X = \{(0, 0), (0, x), (x, 0)\} \subset \mathbb{R}^2$. Let $d = d_\infty |_X$ be the restriction on X of the metric d_∞ given in the proof of the Proposition 1.6. Then $d((0, 0), (x, 0)) = d((0, 0), (0, x) = d((x, 0), (0, x)) = x$. Therefore (X, d) is an ultrametric space. By hypothesis there exists $\varepsilon > 0$ such that $\mathcal{B}_{f \circ d}((0, 0), \varepsilon) = \{(0, 0)\}$. So it follows that $(f \circ d)((0, 0), (x, 0)) = f(d((0, 0), (x, 0))) = f(x) \geq \varepsilon$. We obtain the result from Theorem 3.2

□

Theorem 3.2 is also valid for the case when the dominio of f is a finite product of $[0, \infty)$. Precisely, we obtain the following theorem.

THEOREM 3.4. *Let $f : [0, \infty)^T \longrightarrow [0, \infty)$ be an ultrametric-metric-preserving function where T is finite. Then f is continuous at 0, if and only if for every $\varepsilon > 0$ there exists an $x > 0$ such that $f(x) < \varepsilon$.*

PROOF. We omitted the proof of the necessity. Conversely, let $\varepsilon > 0$. There exists $x \in [0,\infty)^T$ such that $x > 0$ and $f(x) < \frac{\varepsilon}{2}$ $(*)$. We consider the set $U = \bigcap_{t \in T} \pi_t^{-1}([0, \min x_t))$. Clearly, U is an open neighborhood of 0. Let $y \in U$. Then $0 \leq y \leq x$. By Theorem 1.7 and $(*)$, we obtain $f(y) \leq 2f(x) < \varepsilon$. That is, f is continuous at 0.

□

The following example shows that the assumption T *is finite* in Theorem 3.4 can not be ignored.

EXAMPLE 3.5. *Let* $f : [0,\infty)^{\mathbb{N}} \longrightarrow [0,\infty)$ *given by* $f((x_n)) = \sup_{n \in \mathbb{N}} \{\min\{1, x_n\}\}$.

Applying Theorem 1.7, it is easy to see that f is ultrametric-metric-preserving. Now we shall prove that f is not continuous at 0. Let $0 < \varepsilon < 1$ and U be an opened neighborhood of 0 (in the product topology). Then there are $\delta > 0$ and a nonempty finite subset F of $\mathbb{N}$ such that $\bigcap_{n \in F} \pi_n^{-1}([0,\delta)) \subset U$. Define $x = (x_n) \in [0,\infty)^{\mathbb{N}}$ by

$$x_n = \begin{cases} \dfrac{\delta}{2} & \text{for } n \in F; \\ 1 & \text{otherwise}. \end{cases}$$

So $x \in U$ and $f(x) = 1 > \varepsilon$. Thus $f(U) \nsubseteq [0,\varepsilon)$ and we conclude the desired result. Next, it is shown that f satisfies the necessity in Theorem 3.4. Let $\varepsilon > 0$ and n_0 be a positive integer such that $\frac{1}{n_0} < \varepsilon$. Pick the point $x = (x_n) \in [0,\infty)^{\mathbb{N}}$ where $x_n = \frac{1}{n_0}$ for each $n \in \mathbb{N}$. It is obvious that $f(x) = \frac{1}{n_0}$; that is, the mentioned proposition is true.

□

Let Z be a topological space. We say that a function $f : Z \longrightarrow \mathbb{R}$ is *lower semicontinuous* if $f^{-1}(-\infty, a]$ is a closed set for all $a \in \mathbb{R}$.

It is well known that if f is lower semicontinuous and g is continuous, then $f \circ g$ is lower semicontinuous. Thus, we obtain the following result.

PROPOSITION 3.6. *Let (X, d) be a metric space. If $f : [0, \infty) \longrightarrow [0, \infty)$ is lower semicontinuous, then $f \circ d : X \times X \longrightarrow [0, \infty)$ is lower semicontinuous.*

□

PROPOSITION 3.7. *Let $f : [0, \infty) \longrightarrow [0, \infty)$ be an ultrametric-metric-preserving function. If f is continuous at $(0, \infty)$ and discontinuous at 0 then f is lower semicontinuous.*

PROOF. Since f is discontinuous at 0 there exists $\varepsilon > 0$ such that $f(x) > \varepsilon$ for all $x > 0$. Then

$$f^{-1}(-\infty, a] = \begin{cases} \emptyset & \text{for } a < 0; \\ \{0\} & \text{for } a \in [0, \varepsilon]; \\ \{0\} \cup f^{-1}[\varepsilon, a] & \text{for } a > \varepsilon. \end{cases}$$

In each case the set $f^{-1}(-\infty, a]$ is closed. Therefore f is lower semicontinuous.

□

The following example shows a function f which does not satisfies the condition of the continuity in the Proposition 3.7, and however it is lower semicontinuous.

EXAMPLE 3.8. *Let*

$$f(x) = \begin{cases} x & \text{if} \quad x \in [0, 1); \\ x - n + \frac{1}{2} & \text{if} \quad x \in [n, n + \frac{1}{2}), \text{ for each } n \in \mathbb{N}; \\ x - n & \text{if} \quad x \in [n + \frac{1}{2}, n + 1), \text{ for each } n \in \mathbb{N}. \end{cases} \tag{3}$$

f is continuous at $x = 0$ and discontinuous at $x \in \{n, n + \frac{1}{2} : n \in \mathbb{N}\}$. By Theorem 1.5 $f \in \mathcal{UM}$. In fact, let $0 \leq a \leq b$. Just consider the case nontrivial $\frac{1}{2} \leq a$. Due $\mid f(a) - f(b) \mid < \frac{1}{2}$, we obtain $f(a) < f(b) + \frac{1}{2} \leq 2f(b)$ as was desired. It is easy to see that $f \in \mathcal{S}$. On other hand, since $\mathrm{Im}f = [0,1)$, it follows that

$$f^{-1}(-\infty, a] = \begin{cases} [0,a] & \text{if } 0 \le a < \frac{1}{2}; \\ \bigcup_{n\in\mathbb{N}} (A_n \cup B_n) \cup [0,a] & \text{if } \frac{1}{2} \le a < 1, \\ [0,\infty) & \text{if } a \ge 1, \end{cases}$$

where $A_n = [n, a+n-\frac{1}{2}]$ y $B_n = [n+\frac{1}{2}, a+n]$ for each $n \in \mathbb{N}$. Thus f is lower semicontinuous.

□

4 $\mathcal{UM}$ and Convexity

LEMMA 4.1. *If $f : [0,\infty) \longrightarrow [0,\infty)$ is amenable and convex (strictly convex), then the function $\dfrac{f(x)}{x}$ is nondecreasing (increasing) on $(0,\infty)$.*

PROOF. Let $a, b \in (0,\infty)$ and $a < b$. Since f is convex (strictly convex), we obtain

$$\begin{aligned} f(a) &= f\left(\left(1-\frac{a}{b}\right)(0) + \left(\frac{a}{b}\right)(b)\right) \\ &\le (<) \left(1-\frac{a}{b}\right) f(0) + \left(\frac{a}{b}\right) f(b) \\ &= \left(\frac{a}{b}\right) f(b). \end{aligned}$$

Therefore $\dfrac{f(a)}{a} \le (<) \dfrac{f(b)}{b}$, as required. □

Given a function $f : X \longrightarrow X$, we denote by Fix_f the set of all fixed points of f.

COROLLARY 4.2. *Let $f : [0,\infty) \longrightarrow [0,\infty)$ amenable and convex. If $f(a) = a$ and $f(b) = b$ where $0 < a < b$, then $[a,b] \subset Fix_f$.*

PROOF. Let $x \in (a,b)$. By Lemma 4.1

$$1 = \frac{f(a)}{a} \le \frac{f(x)}{x} \le \frac{f(b)}{b} = 1.$$

Hence $f(x) = x$ as asserted.

□

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THEOREM 4.3. *Let $f : [0, \infty) \longrightarrow [0, \infty)$ be amenable. If f is strictly convex, then f is ultrametric-metric-preserving and is not metric-preserving.*

PROOF. Let $0 \leq a < b$. We shall prove that f is increasing. First observe that if $a = 0$, then $f(0) = 0 < f(b)$. Now suppose $0 < a$. By Lemma 4.1 the function $\dfrac{f(x)}{x}$ is increasing on $(0, \infty)$. Therefore $\dfrac{f(a)}{a} < \dfrac{f(b)}{b}$. So $f(a) < \dfrac{a}{b} f(b) < f(b)$. By Theorem 1.5, f is ultrametric-metric-preserving. Furthermore, also of the strictly convexidad of f, it follows the inequality $f\left(\dfrac{a+b}{2}\right) < \dfrac{f(a+b)}{2}$, whence $f\left(\dfrac{a+b}{2}\right) + f\left(\dfrac{a+b}{2}\right) < f(a+b)$, which violates subadditivity. Since $\mathcal{M} \subset \mathcal{S}$, we conclude that f is not metric-preserving.

□

Concerning the previous theorem, it is easy to prove that if we substitute *strictly convex* rather than *convex* in the hypothesis, then the conclusion is $f \in \mathcal{UM}$. In this case the function f may be metric-preserving; for example $f(x) = x$.

The function f given in Example 2.2 is not convex and $f \in \mathcal{UM}$. So, $\mathcal{UM}$ properly contains the whole set of convex functions.

We will need the next result which surely appears in the literature.

LEMMA 4.4. *Given $f : [0, \infty) \longrightarrow [0, \infty)$.*

(1) *if f is amenable and convex then $f(\frac{x}{2^n}) \leq \frac{f(x)}{2^n}$ for all $x \geq 0$ and $n \in \mathbb{N}$,*

(2) *if f is subadditive then $f(\frac{x}{2^n}) \geq \frac{f(x)}{2^n}$ for all $x \geq 0$ and $n \in \mathbb{N}$.*

□

LEMMA 4.5. *If f is ultrametric-metric-preserving and subadditive, then*

$$\frac{x}{f(x)} \leq 2^2 \frac{y}{f(y)} \quad \textit{whenever} \quad 0 < x \leq y.$$

PROOF. Let $0 < x \leq y$. Pick $n \in \mathbb{N}$ so that $2^{n-1} \leq \frac{y}{x} < 2^n$. Since $\frac{y}{2^n} < x$ by Lemma 4.4 (2) and Theorem 1.5

$$\frac{f(y)}{2^n} \leq f\left(\frac{y}{2^n}\right) \leq 2f(x).$$

Hence, and from $\frac{2^{n-1}}{y} \leq \frac{1}{x}$, we obtain

$$\frac{f(y)}{y} \leq 2^{n+1}\frac{f(x)}{y} \leq 2^2\frac{f(x)}{x}.$$

Thus, the result follows.

□

THEOREM 4.6. *Suppose f is amenable, subadditive and convex. Then f is linear.*

PROOF. Since f is convex, by Lemma 4.1 $g(x) = \frac{f(x)}{x}$ is nondecreasing on $(0, \infty)$. We will prove that g is a constant function. Let $a, b \in (0, \infty)$, with $a < b$. We know $g(a) \leq g(b)$. On the other hand, we choose $n \in \mathbb{N}$ so that $\frac{b}{2^n} < a$. By Lemma 4.4, and again by Lemma 4.1, it follows

$$g(b) = \frac{f(b)}{b} = \frac{\frac{f(b)}{2^n}}{\frac{b}{2^n}} = \frac{f\left(\frac{b}{2^n}\right)}{\frac{b}{2^n}} \leq \frac{f(a)}{a} = g(a).$$

Thus $g(a) = g(b)$. We conclude that $\frac{f(x)}{x} = m$, where $m = \frac{f(a)}{a}$ for any $a > 0$. So, $f(x) = mx$ for all $x \in [0, \infty)$, as required. This completes the proof.

□

5 $\mathcal{UM}$ and Differentiability

We know from [3], that for any metric-preserving function f, $f'(0)$ exists in the extended sense. This statement is based on whether the set $K_f = \{r > 0 : f(x) \leq rx, \text{ for all } x \geq 0\}$ is empty. Namely

$$(a) \text{ if } K_f = \emptyset \text{ then } f'(0) = \infty; \qquad (b) \text{ if } K_f \neq \emptyset \text{ then } f'(0) < \infty. \tag{4}$$

By the following two examples, it is shown that the propositions (a) y (b) are false when f is ultrametric-metric-preserving and continuous at $x = 0$.

EXAMPLE 5.1. *Let*

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \cap [0, \infty); \\ \dfrac{x^2}{2} & \text{if } x \in \mathbb{Q}^c \cap [0, \infty). \end{cases} \tag{5}$$

It is easy to see that $K_f = \emptyset$ and $f'(0) = 0$. By Proposition 2.4, $f \in \mathcal{UM}$ and from Theorem 3.1, $f \notin \mathcal{M}$.

□

EXAMPLE 5.2. *Let* $f : [0, \infty) \longrightarrow [0, \infty)$ *be given by*

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \cap [0, 1]; \\ \dfrac{x}{2} & \text{if } x \in \mathbb{Q}^c \cap [0, 1]; \\ \dfrac{1}{2} & \text{if } x > 1. \end{cases} \tag{6}$$

From the Theorem 3.1 $f \notin \mathcal{M}$. We observe that $f(x) \in [g(x), 2g(x)]$, where

$$g(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \in [0, 1]; \\ \dfrac{1}{2} & \text{if } x > 1. \end{cases}$$

Thus, by Proposition 2.4 $f \in \mathcal{UM}$. We claim that $\lim\limits_{x \longrightarrow 0} \dfrac{f(x)}{x}$ does not exist. In fact, let t be a sequence that converges to 0. Then $\left\{\dfrac{f(t)}{t}\right\} \longrightarrow 1$ whenever

$t \subset \mathbb{Q}$, but $\left\{\dfrac{f(t)}{t}\right\} \longrightarrow \frac{1}{2}$ for $t \subset \mathbb{Q}^c$. That is, $f'(0)$ does not exist. Observe that $f(x) \leq x$ for every $x \in [0,\infty)$, so $K_f \neq \emptyset$.

□

THEOREM 5.3. *Suppose f is ultrametric-metric-preserving, subadditive and $K_f = \emptyset$. Then $f'(0) = \infty$.*

PROOF. Let $n \in \mathbb{N}$. Since $K_f = \emptyset$, we can to choose $y > 0$ such that $2^2 ny \leq f(y)$. Let $x \in (0, y]$. By Lemma 4.5 and the above inequality

$$\frac{x}{f(x)} \leq 2^2 \frac{y}{f(y)} \leq \frac{1}{n}.$$

Thus $\dfrac{f(x)}{x} \geq n$. This completes the proof.

□

The function f in Example 5.1 satisfies $f \in \mathcal{UM}$, $K_f = \emptyset$, $f \notin \mathcal{S}$ and $f'(0) = 0$. Accordingly, we conclude that the hypothesis $f \in \mathcal{S}$ in Theorem 5.3 is essential.
Next example shows a ultrametric-metric-preserving function and continuous on $[0,\infty)$, which is not differentiable at 0.

EXAMPLE 5.4. *Consider the continuous function f defined as*

$$f(x) = \begin{cases} 0 & \text{if } x = 0; \\ \\ x + x \mid \sin \frac{1}{x} \mid & \text{if } x > 0. \end{cases} \tag{7}$$

Since $f(x) \in [x, 2x]$ for all $x \geq 0$, by the Proposition 2.4, $f \in \mathcal{UM}$. Also, we have $K_f \neq \emptyset$. On the other hand it is clear that$f'(0)$ does not exist.

□

Let us observe that the function in Example 5.4 is not subadditive; indeed $f(\frac{1}{\pi} + \frac{2}{\pi}) > f(\frac{1}{\pi}) + f(\frac{2}{\pi})$, and the function (2) is continuous on $[0,\infty)$, ultrametric-metric-preserving, subadditive and differentiable at 0. The proof of the next result is identical to the verification of the equality (4.2) $f'(0) = \min K_f$ which appears in [3].

THEOREM 5.5. *Let f be ultrametric-metric-preserving, continuous and subadditive. If $K_f \neq \emptyset$ then $f'(0) < \infty$.*

□

It is known that every metric function f such that $f'(0) < \infty$, it is differentiable almost everywhere [7]. Actually it is unknown if this result is valid for ultrametric-metric-preserving functions. We leave this problem for the interested reader and also the study the behavior of ultrametric-metric-preserving functions with $f'(0) = \infty$.

Conclusions

We note that the continuity at 0 of a ultrametric-metric-preserving function f is equivalent to give an ultrametric d, such that $f \circ d$ becomes a discrete uniformly metric. A possible direction of this research is the study of interesting global properties of f which they can be deduced from the behaviors of f at 0, such as continuity at 0 or the value of $f'(0)$ (finite or infinite) if it exists. Moreover, also as an additional study it is the characterization of the set of fixed points of an ultrametric-metric-preserving function f, when f is not necessarily strictly convex.

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