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Level sets as boundaries and a topological application of the gradient of a differentiable function

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Abstract
En este artículo, se demuestra que para un conjunto de nivel no vacío R_k, $k \in \mathbb{R}$, asociado a una funcion diferenciable $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, los puntos regulares no doblemente aislados contenidos en R_k son puntos frontera de los conjuntos abiertos $R_{<k}$ y $R_{>k}$, los cuales corresponden a las uniones disjuntas (izquierda o derecha) que contienen los demás conjuntos de nivel que son complementarios a R_k. Además, se prueba que los puntos críticos doblemente aislados son puntos frontera de solo uno de estos conjuntos abiertos complementarios. La propiedad de ser frontera se deriva únicamente usando la estructura diferenciable del mapeo f que no necesariamente es de clase C^1; en particular, se utilizan propiedades del gradiente y el comportamiento de f cerca de los puntos regulares o bien de los puntos críticos. Finalmente, se proporcionan fórmulas para comparar los conjuntos interior, frontera y exterior de $R_{\leq k}$ y $R_{\geq k}$, con los conjuntos correspondientes a $R_{<k}$ y $R_{>k}$.

In this paper we prove that, given a non empty level set R_k, $k \in \mathbb{R}$, for a differentiable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, regular points and non doubly isolated critical points contained in R_k are simultaneously boundary points of the open sets $R_{<k}$ and $R_{>k}$, which correspond to the disjoint union (either left or right) containing the complementary level sets to R_k. Moreover, we prove that doubly isolated critical points are boundary points of only one of them. The property to be boundary is only derived by using the differentiable structure of f which is not necessarily a C^1 map, in particular, we use properties of gradient and the behavior of f around either regular or critical points. Finally, relating formulas are given to compare interiors, boundaries and exteriors of $R_{\leq k}$ and $R_{\geq k}$, with the corresponding to $R_{<k}$ and $R_{>k}$.

Keywords: Differentiable function, Critical points, Isolated points, Level sets, Boundaries

1. Introduction

Let n be a positive integer, $k \in \mathbb{R}$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a map defined in the Euclidean topological space $\mathbb{R}^n$. A common way to deal with regions of $\mathbb{R}^n$ is by means of either

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inequalities or equalities, which may be used to define explicitly such regions,

$$\begin{array}{ll} \text{a) } R_{\geq k} := \{\vec{p} \in \mathbb{R}^n : f(\vec{p}) \geq k\}, & \text{b) } R_{\leq k} := \{\vec{p} \in \mathbb{R}^n : f(\vec{p}) \leq k\}, \\ \text{c) } R_{>k} := \{\vec{p} \in \mathbb{R}^n : f(\vec{p}) > k\}, & \text{d) } R_{<k} := \{\vec{p} \in \mathbb{R}^n : f(\vec{p}) < k\}, \\ \text{e) } R_{\neq k} := \{\vec{p} \in \mathbb{R}^n : f(\vec{p}) \neq k\}, & \text{f) } R_k := \{\vec{p} \in \mathbb{R}^n : f(\vec{p}) = k\}. \end{array} \quad (1)$$

The case of equality is usually known as a *level set* in $\mathbb{R}^n$. Moreover, the fact that $\mathbb{R}^n$ is the domain of f, implies that we may stratify the space $\mathbb{R}^n$ by means of the level sets of f,

$$\mathbb{R}^n = R_{<k} \cup R_k \cup R_{>k}.$$

Sometimes one needs to study the topological behavior of these regions considering the topological structure of $\mathbb{R}^n$. If f is continuous and $\mathbb{R}^n$ has the usual topology, one can say that the regions a), b) and f) in (1) are closed subsets in $\mathbb{R}^n$, and regions c), d) and e) are open. Although these last facts are clear, some topological properties related with these sets are not so easy to identify. This is the case for limit points which are relevant in analytical processes. For instance, if given a subset M of $\mathbb{R}^n$, one needs to establish its derived set M', one has to calculate the closure $\overline{M}$ of M, the set of isolated points $\operatorname{Iso}(M)$ of M, and to take the relative difference $M' = \overline{M} - \operatorname{Iso}(M)$. By the way, the sets $\overline{M}$ and $\operatorname{Iso}(M)$ are closely related with the boundary set ∂M of M, since $\overline{M} = M \cup \partial M$ and $\operatorname{Iso}(M) \subset \partial M$. Hence, in this approach we are dealing with the boundary set of M, and therefore notice that the boundary ∂M also provides us with the interior M° and the exterior $\operatorname{Ext}(M)$ of M, since $M^\circ = \overline{M} - \partial M$ and $\operatorname{Ext}(M) = \mathbb{R}^n - \overline{M} = \overline{M}^c$ (see [4]). We mention at this moment that for topological purposes, our principal interest in a subset M of $\mathbb{R}^n$ is to determine the boundary set ∂M and the isolated set $\operatorname{Iso}(M)$ of M.

Coming back to the sets given by the relations (1), we may assert that the boundary of fifth and sixth regions is the sixth one of them. It seems to be that the level set R_k satisfies to be the common boundary of all the regions given in (1); but this is not always the case for first, second, third and fourth regions. Consider the following case. Let

$$F(x,y) = x(x+7)^2 - (x+2y-5)^2,$$
$$B_1 = \left\{(x,y) : x(x+7)^2 - (x+2y-5)^2 > 0\right\}$$

and

$$A_1 = \left\{(x,y) : x(x+7)^2 - (x+2y-5)^2 = 0\right\}.$$

In this occasion, A_1 has an isolated point given by $\vec{p}_1 = (-7,6)$ which is *not* a boundary point for B_1. It is of interest that, in contrast, the "complementary" open region given by

$$C_1 = \left\{(x,y) : x(x+7)^2 - (x+2y-5)^2 < 0\right\}$$

has whole A_1, including the isolated point, as its boundary. See the figure 1.

Due to these negative cases, it is important to consider the analytical character of a point in the level set. For the above example, we have that the gradient of F, ∇F, vanishes in $(-7,6)$. In other words, $(-7,6)$ is a critical point for F. But, is it always the same situation for all the critical points of a scalar field which are in the

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corresponding level set? Again, the answer is *negative*. To see this, just take a look of the following examples,

$$
\begin{aligned}
G(x,y) &= (x+10)(x+2)^2 - (x+2y-5)^2, \\
\vec{p}_2 &= (-2, 7/2), \\
A_2 &= \left\{(x,y) : (x+10)(x+2)^2 - (x+2y-5)^2 = 0\right\}, \\
B_2 &= \left\{(x,y) : (x+10)(x+2)^2 - (x+2y-5)^2 > 0\right\}, \\
C_2 &= \left\{(x,y) : (x+10)(x+2)^2 - (x+2y-5)^2 < 0\right\},
\end{aligned} \tag{2}
$$

and

$$
\begin{aligned}
H(x,y) &= (x+4)^3 - (x+2y-5)^2, \\
\vec{p}_3 &= (-4, 9/2), \\
A_3 &= \left\{(x,y) : (x+4)^3 - (x+2y-5)^2 = 0\right\}, \\
B_3 &= \left\{(x,y) : (x+4)^3 - (x+2y-5)^2 > 0\right\}, \\
C_3 &= \left\{(x,y) : (x+4)^3 - (x+2y-5)^2 < 0\right\}.
\end{aligned} \tag{3}
$$

See figures 2 and 3. In both examples, the critical points are inside the level set and they are boundary points of the two corresponding open regions.

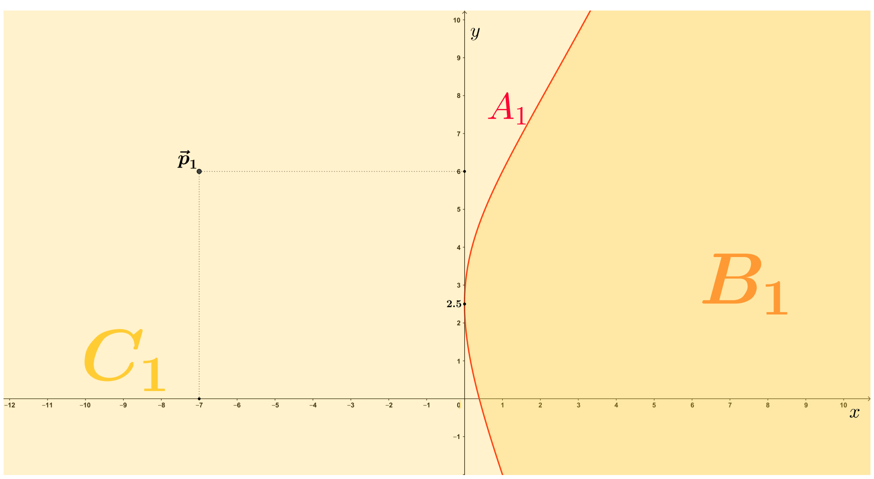

Figure. 1

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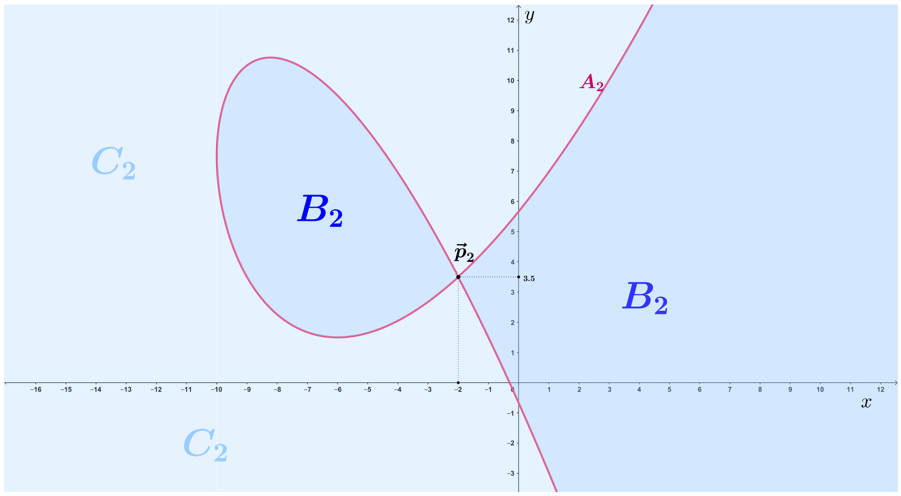

Figure. 2

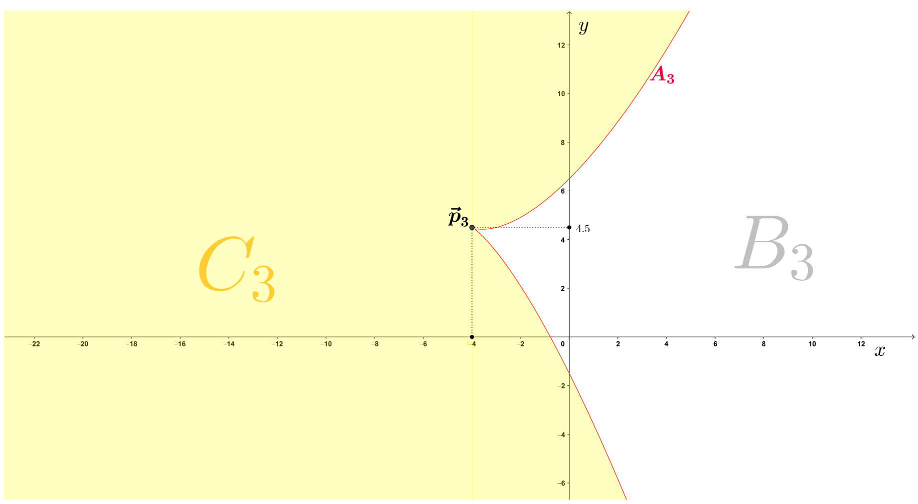

Figure. 3

The aim of this paper is to show that, on one hand, the regular and non isolated critical points are boundary points of both $R_{>k}$ and $R_{<k}$, and on the other hand, when the dimension n is greater than or equals to 2, the isolated critical points are boundary points of either $R_{>k}$ or $R_{<k}$ but not both. We may precise the last statement by stating that if a point $\vec{p}$ in R_k is isolated then either $\vec{p}$ is a boundary point of $R_{>k}$ if and only if $f(\vec{p}) = k$ is a local minimum, or $\vec{p}$ is a boundary point of $R_{<k}$ if and only if $f(\vec{p}) = k$ is a local maximum. That is we prove the following main theorem.

Theorem 1. Assume $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a differentiable map and $A \neq \emptyset$, then the

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following statements holds.

1. The regular points of f which are in R_k are boundary points both of $R_{>k}$ and of $R_{<k}$.
2. The isolated critical points of f which are in R_k and are not doubly isolated, are boundary points both of $R_{>k}$ and of $R_{<k}$.
3. If $n \geq 2$, the isolated points of R_k are doubly isolated critical points of R_k, and they are boundary points of either $R_{>k}$ or $R_{<k}$ but not both. Symbolically,

$$\operatorname{Iso}(R_k) = (\operatorname{Iso}(R_k) \cap \partial R_{>k}) \cup (\operatorname{Iso}(R_k) \cap \partial R_{<k}),$$

and

$$\operatorname{Iso}(R_k) \cap \partial R_{>k} \cap \partial R_{<k} = \emptyset.$$

Moreover, if $\vec{a}$ is an isolated point of R_k, then we have the following: $\vec{a}$ is a boundary point of $R_{>k}$ if and only if $f(\vec{a}) = k$ is a local minimum; $\vec{a}$ is a boundary point of $R_{<k}$ if and only if $f(\vec{a}) = k$ is a local maximum.

As an immediate result we obtain the Corollary 2 which states that when the only critical points of f which are inside R_k are isolated but non doubly isolated, it holds that

$$R_{>k} \neq \emptyset \neq R_{<k}, \qquad \partial R_{<k} = R_k = \partial R_{>k}. \tag{4}$$

When one preserves the condition of isolation of the critical points but now we consider the regions given by the non strict inequalities $R_{\geq k}$ and $R_{\leq k}$, one can prove the following main theorem, which states that there is an obstruction to have the analogous of equation (4).

Theorem 2. Assuming $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a differentiable map, $n \geq 2$, and that the only critical points of f which are in R_k are isolated, then we have the following facts:

$$\begin{aligned} R^\circ_{\geq k} &= R_{>k} \cup (\operatorname{Iso}(R_k) \cap \partial R_{>k}) = R_{>k} \cup (\operatorname{Iso}(R_k) - \partial R_{<k}), \\ \partial R_{\geq k} &= R_k - (\operatorname{Iso}(R_k) \cap \partial R_{>k}) = R_k - (\operatorname{Iso}(R_k) - \partial R_{<k}), \\ \operatorname{Ext}(R_{\geq k}) &= R_{<k}, \end{aligned} \tag{5}$$

$$\begin{aligned} R^\circ_{\leq k} &= R_{<k} \cup (\operatorname{Iso}(R_k) \cap \partial R_{<k}) = R_{<k} \cup (\operatorname{Iso}(R_k) - \partial R_{>k}), \\ \partial R_{\leq k} &= R_k - (\operatorname{Iso}(R_k) \cap \partial R_{<k}) = R_k - (\operatorname{Iso}(R_k) - \partial R_{>k}), \\ \operatorname{Ext}(R_{\leq k}) &= R_{>k}. \end{aligned} \tag{6}$$

Notice that while Corollary 2 gives enough conditions to have that R_k is the common boundary of $R_{>k}$ and $R_{<k}$, Theorem 2 gives more information since it involves not only the boundaries but the corresponding interior and exterior sets and contemplate the cases where the isolated critical points are also topologically isolated (see figure 1).

We want to point out that we do not include the case $n = 1$ because the doubly isolated points p make a disconnection on the open intervals $B(p)$ which are neighborhood of p in the sense that $B(p) - \{p\}$ is not connected while for $n \geq 2$ it is not

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the case, we use this last fact on our results as one can see in the corresponding proofs (see the proof of Theorems 1 and 2). As one may check, if $f(x) = x^3, x \in \mathbb{R}$,

$$A = \{0\}, \quad B = \mathbb{R}_+, \quad C = \mathbb{R}_-,$$

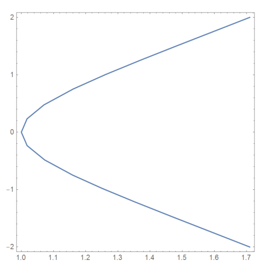
$\partial C = A = \partial B$ but in our result 3) of Theorem 1 the statement is that A is boundary of only one of them B or C, which does not holds in this example.

Notice that our results are given when the domain of the map f is whole $\mathbb{R}^n$ instead of only an open proper subset Ω, for instance, consider the cusp $f(x, y) = y^2 - x^3$. In this example,

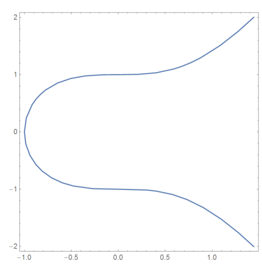
$$R_{<k} = \{(x, y) : y^2 < x^3 + k\}, \quad R_k = \{(x, y) : y^2 = x^3 + k\},$$

$$R_{>k} = \{(x, y) : y^2 > x^3 + k\}$$

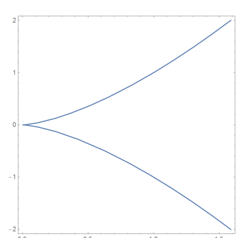
hence if $k < 0$, then each point in $R_{<k} \cup R_k$ is a regular point while $R_{>k}$ has the only one point $(0, 0)$ as critical point. One observes that if the map $f(x, y)$ is restricted to a proper open subset $\Omega \subset \mathbb{R}^2$, in $R_{>k}$ it may occur that eventually the critical point does not belong to Ω.



R_k, $k < 0$



R_k, $k > 0$



R_k, $k = 0$

Figure. 4

Another point to note is that the above examples consider two kind of critical points, the isolated and the non isolated in topological sense. Hence, these examples (see Figures 2 and 3) also show that we will deal with differentiable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that their critical points are not necessarily non degenerate (recall from calculus that a critical point is non degenerate if the determinant of the second derivative at such critical point is different from zero, see [1]) and therefore we are not interested in using proof techniques which consider second derivative criterion. In this sense, the mathematical tools used in our proofs are classical and hence elementary by considering that the map f is differentiable but not necessarily a C^1 function (see

Proof of (1) in Theorem 1). However, if f is a C^ℓ function, for instance $\ell = 2$ and $n = 2$, one obtains a well known classification of the critical points inside a level set; to do that, given a critical point $\vec{x}$ in R_k one uses the determinant of the Hessian matrix of f at $\vec{x}$, $H(f)_{\vec{x}}$, in order to get the basic types of singular points: If $H(f)_{\vec{x}} > 0$, $\vec{x}$ is an isolated point; if $H(f)_{\vec{x}} < 0$, $\vec{x}$ is a double point; if $H(f)_{\vec{x}} = 0$, $\vec{x}$ is either a cusp of the first or second kind, or an isolated point or a tacnode (see [2, Sec. 15]). In our proofs we do not use these kind of results.

Certainly, there are very ample theories that develop the comprehensive study of the level sets associated to a differentiable map f between more general objects called manifolds but the corresponding results are heavily supported on the assumption that the map f is actually of class C^ℓ, for at least $\ell \geq 1$ (see [3]) which, effectively, gives stronger results which involve nice descriptions of the topological behavior of the level sets of f, for instance, we may look at the Intersection Numbers, Euler Characteristic and Morse Theory (see [3]).

2. Notation

We will recall standard notation (see [4] and [1]). Let n be a positive integer, $\mathbb{R}$ be the field of the real numbers, and $\mathbb{R}^n$ be the euclidean space of dimension n over $\mathbb{R}$, with the standard inner product given in the following way: if $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{R}^n$, then

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = a_1 b_1 + \ldots + a_n b_n.$$

Since this inner product is positive definite and non degenerated, we may establish the canonical norm as usual,

$$\|(a_1, \ldots, a_n)\| = \left[(a_1, \ldots, a_n) \cdot (a_1, \ldots, a_n)\right]^{\frac{1}{2}}.$$

Also, remember Cauchy-Schwarz's inequality,

$$|(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n)| \leq \|(a_1, \ldots, a_n)\| \, \|(b_1, \ldots, b_n)\|,$$

and that the equality holds only when the vectors are linearly dependent; which, for non zero vectors, means that they are parallel. By the way, we represent the vectors with top arrow letters; that is, $\vec{a} = (a_1, \ldots, a_n)$. Continuing, using the norm we define the distance between vectors as usual,

$$\mathrm{d}(\vec{a}, \vec{b}) = ||\vec{b} - \vec{a}||.$$

Besides, this metric allows us to establish the canonical topology for $\mathbb{R}^n$ generated by the open balls $B(\vec{a}; r)$, with $r > 0$, which are defined by

$$B(\vec{a}; r) = \{\vec{p} : \|\vec{p} - \vec{a}\| < r\}.$$

With respect to $\mathbb{R}$, the open ball corresponds with the open interval $B(a; r) = (a - r; a + r)$. We may consider also the infinite intervals given by

$$(a; +\infty) = \{x : a < x\}, \quad (-\infty; a) = \{x : x < a\},$$
$$[a; +\infty) = \{a\} \cup (a; +\infty), \quad (-\infty; a] = (-\infty; a) \cup \{a\}.$$

Since the usual topology of $\mathbb{R}^n$ is generated by the open balls, we have to say that a subset of $\mathbb{R}^n$ is open if it is the union of open balls. A closed subset of $\mathbb{R}^n$ is

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the complement of an open subset of $\mathbb{R}^n$. By the way, if S is a subset of $\mathbb{R}^n$, its complement is denoted by S^c. Now, let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a map. We say that f is continuous if for every open subset T of $\mathbb{R}$ we have that its preimage $f^{-1}(T)$ is an open subset of $\mathbb{R}^n$. Let $\vec{a} \in \mathbb{R}^n$. We say that f is differentiable in $\vec{a}$ if there exist $r > 0$ and a map $\mathcal{E} : B(\vec{0}; r) \longrightarrow \mathbb{R}$ continuous in $\vec{0}$ such that, for every $\vec{b} \in B(\vec{a}; r)$, it holds that

$$f(\vec{b}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{b} - \vec{a}) + ||\vec{b} - \vec{a}||\mathcal{E}(\vec{b} - \vec{a})$$

where

$$\lim_{\vec{b} \longrightarrow \vec{a}} \mathcal{E}\big(\vec{b} - \vec{a}\big) = \mathcal{E}(\vec{0}) = 0.$$

The above formula is called the Taylor's formula of first order for f about $\vec{a}$ (see [1]).

Finally, we recall the definition of regular and critical points. Let $S \subset \mathbb{R}^n$, $\vec{a}$ be an inner point of S, and $f : S \longrightarrow \mathbb{R}$ be a map.

Definition 1. Assume f is differentiable at $\vec{a}$, and let $T \subset S$, $\vec{a} \in T$. The point $\vec{a}$ is

1. a *regular point of* f if $\nabla f(\vec{a}) \neq \vec{0}$;
2. a *critical point of* f if $\nabla f(\vec{a}) = \vec{0}$;
3. an *isolated critical point of* f if $\vec{a}$ is a critical point of f and if there exists $r > 0$ such that, in $B(a; r)$, $\vec{a}$ is the only critical point of f;
4. a *doubly isolated critical point of* T if $\vec{a}$ is an isolated critical point of f and an isolated point of T.

3. The results

3.1 Topological properties for level sets of continuous maps

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous map. In order to simplify the notation along our proofs we choose

$$B = R_{>k}, \qquad A = R_k, \qquad C = R_{<k}$$

according to (1).

Lemma 1. Considering the usual topology of $\mathbb{R}$ and $\mathbb{R}^n$, we have the following properties:

1. **Disjunction.** A, B and C are pairwise disjoint.
2. **Completeness.** $A \cup B \cup C = \mathbb{R}^n$.
3. **Typology.** A is a closed subset and B and C are open subsets of $\mathbb{R}^n$.
4. **Separatization.** B and C are separate.
5. **Boundary.** $\partial B \subset A$ and $\partial C \subset A$.
6. **Emptiness in level sets.** If $A = \emptyset$, then some of the subsets B or C is also empty and the other one is $\mathbb{R}^n$.

Proof. Notice that the proof of 1 and 2 are straightforward. Also, 3 follows directly from the continuity of f. We will proceed to prove the other statements.

4. (Separatization) It holds that $A \cup B = f^{-1}([k;+\infty))$ because

$$A \cup B = f^{-1}(\{k\}) \cup f^{-1}((k;+\infty)) = f^{-1}(\{k\} \cup (k;+\infty)) = f^{-1}([k;+\infty)).$$

Since $[k;+\infty)$ is a closed subset and since f is continuous, we have that $A \cup B$ is a closed subset. Analogously, we may prove that $A \cup C$ is a closed subset. Hence, in particular, $\overline{B} \subset A \cup B$ and $\overline{C} \subset A \cup C$. Besides, by 1 and 2 of this theorem, we have that $A \cup B = C^c$ and $A \cup C = B^c$. So, $\overline{B} \subset C^c$ and $\overline{C} \subset B^c$. Even better, $\overline{B} \cap C = \emptyset$ and $\overline{C} \cap B = \emptyset$; whence B and C are separated.

5. (Boundary) Since B and C are separate, we have that $\partial B \subset C^c = A \cup B$ and $\partial C \subset B^c = A \cup C$. Now, since B and C are open subsets, it happens that $\partial B \cap B = \emptyset$ and $\partial C \cap C = \emptyset$. From these two equalities and the previous inclusions, we conclude that $\partial B \subset A$ and $\partial C \subset A$.

6. (Emptiness in level sets) Let us assume that $A = \emptyset$. From 1, we have that $B \cup C = \mathbb{R}^n$. Proceeding by contradiction, assume that $B \neq \emptyset$ and $C \neq \emptyset$. From 4, it holds now that B and C conform a separation for $\mathbb{R}^n$. But this is an absurd, taking into account that $\mathbb{R}^n$ is connected. Hence, either $B = \emptyset$ or $C = \emptyset$; but not both of them, since $B \cup C = \mathbb{R}^n$. Finally, if, for instance, $B = \emptyset$, then

$$C = \emptyset \cup C = A \cup B \cup C = \mathbb{R}^n.$$

That is, generalizing, some of the subsets B or C is empty and the other one is $\mathbb{R}^n$.
■

3.2 Regular points and isolated critical points in level sets of differentiable maps

In this subsection $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a differentiable map and A, B, C are as before. Assume that $A \neq \emptyset$. We will prove the main Theorem 1.

3.2.1 Proof of main Theorem 1

Without loss of generality, we choose $k = 0$; this enables us to simplify the terminology considerably. Let $\vec{p}_1 \in A$; thus $f(\vec{p}_1) = 0$.

We prove 1. Let us assume that $\vec{p}_1$ is a regular point. Hence, $\nabla f(\vec{p}_1) \neq \vec{0}$. First, we will prove that the subsets B and C are not empty. Indeed, let $\vec{u} = \nabla f(\vec{p}_1) / \|\nabla f(\vec{p}_1)\|$ and define the straight line through the point $\vec{p}_1$

$$\mathcal{L} = \{\vec{v} \in \mathbb{R}^n : \text{there exists} \quad t \in \mathbb{R} \quad \text{such that} \quad \vec{v} = \vec{p}_1 + t\vec{u}\},$$

which is normal to A at $\vec{p}_1$. We claim that there exists $\delta > 0$ such that

$$B(\vec{p}_1; \delta) \cap (A \cap \mathcal{L}) = \{\vec{p}_1\}; \tag{7}$$

that is, in the open ball $B(\vec{p}_1; \delta)$, it happens that A and $\mathcal{L}$ only share $\vec{p}_1$ as a common point. To prove this claim, let us assume, by contradiction, that for every $\delta > 0$, there exists $\vec{q} \in B(\vec{p}_1;\delta) \cap (A \cap \mathcal{L})$ such that $\vec{q} \neq \vec{p}_1$. So, $0 < \|\vec{q} - \vec{p}_1\| < \delta$, $f(\vec{q}) = 0$ and $\vec{q} \in \mathcal{L}$. On using Taylor's formula of first order for f about $\vec{p}_1$, we have that

$$f(\vec{q}) = f(\vec{p}_1) + \nabla f(\vec{p}_1) \cdot (\vec{q} - \vec{p}_1) + \|\vec{q} - \vec{p}_1\| \mathcal{E}(\vec{q} - \vec{p}_1) \tag{8}$$

and

$$\lim_{\vec{q} \to \vec{p}_1} \mathcal{E}(\vec{q} - \vec{p}_1) = \mathcal{E}(\vec{0}) = 0.$$

Substituting $f(\vec{p}_1) = 0 = f(\vec{q})$ in (8) we obtain

$$0 = \nabla f(\vec{p}_1) \cdot (\vec{q} - \vec{p}_1) + \|\vec{q} - \vec{p}_1\| \mathcal{E}(\vec{q} - \vec{p}_1).$$

Even better,

$$\nabla f(\vec{p}_1) \cdot (\vec{q} - \vec{p}_1) = -\|\vec{q} - \vec{p}_1\| \mathcal{E}(\vec{q} - \vec{p}_1). \tag{9}$$

Since $\vec{q} \in \mathcal{L}$, it holds that $\nabla f(\vec{p}_1)$ and $\vec{q} - \vec{p}_1$ are *parallel*, hence Cauchy-Schwarz's inequality implies that

$$|\nabla f(\vec{p}_1) \cdot (\vec{q} - \vec{p}_1)| = \|\nabla f(\vec{p}_1)\| \|\vec{q} - \vec{p}_1\|.$$

Using this equality in (9) we get

$$\|\nabla f(\vec{p}_1)\| \|\vec{q} - \vec{p}_1\| = |\nabla f(\vec{p}_1) \cdot (\vec{q} - \vec{p}_1)| = \|\vec{q} - \vec{p}_1\| |\mathcal{E}(\vec{q} - \vec{p}_1)|,$$

and taking into account that $\|\vec{q} - \vec{p}_1\| > 0$, we may cancel out this factor in both sides of the previous chain of equalities obtaining

$$\|\nabla f(\vec{p}_1)\| = |\mathcal{E}(\vec{q} - \vec{p}_1)|.$$

But then, taking the limit as $\vec{q} \longrightarrow \vec{p}_1$ we have that

$$\|\nabla f(\vec{p}_1)\| = \lim_{\vec{q} \longrightarrow \vec{p}_1} \|\nabla f(\vec{p}_1)\| = \lim_{\vec{q} \longrightarrow \vec{p}_1} |\mathcal{E}(\vec{q} - \vec{p}_1)| = 0.$$

Whence $\|\nabla f(\vec{p}_1)\| = 0$, and by this $\nabla f(\vec{p}_1) = \vec{0}$, which is a contradiction since $\vec{p_1}$ is a regular point and the claim (7) is proved. Consequently

$$f(\vec{p}_1 + t\vec{u}) \neq 0, \quad \text{if} \quad 0 < |t| < \delta \tag{10}$$

since $\vec{p}_1 + t\vec{u} \notin A$ for all $0 < |t| < \delta$.

Let us define

$$\mathcal{M} = \{\vec{v} \in \mathbb{R}^n : \text{there exists} \quad t \in (0;\delta) \quad \text{such that} \quad \vec{v} = \vec{p}_1 + t\vec{u}\},$$
$$\mathcal{N} = \{\vec{v} \in \mathbb{R}^n : \text{there exists} \quad t \in (-\delta;0) \quad \text{such that} \quad \vec{v} = \vec{p}_1 + t\vec{u}\}.$$

With respect to these sets, there are several facts which are immediately verifiable and that they are convenient to emphasize for future references. First, we have that $\mathcal{M} \neq \emptyset$ and $\mathcal{N} \neq \emptyset$ since these sets are the homeomorphic images of the open intervals $(0;\delta)$ and $(-\delta;0)$, respectively, under $\vec{h}(t) = \vec{p}_1 + t\vec{u}$. Second, $\mathcal{M} \cup \{\vec{p}_1\} \cup \mathcal{N} = B(\vec{p}_1;\delta) \cap \mathcal{L}$. Besides, no matter if $\vec{v} \in \mathcal{M}$, or if $\vec{v} \in \mathcal{N}$, one obtains from (10) that $f(\vec{v}) \neq 0$. Finally, since the sets $\mathcal{M}$ and $\mathcal{N}$ are connected and f is continuous and different from zero on them, it happens that, applying the intermediate value's theorem to $f(\vec{h}(t))$, the sign of f remains the same when it evaluates at the points of $\mathcal{M}$ or $\mathcal{N}$.

The immediate objective is to prove that f changes its sign when it passes from $\mathcal{M}$ to $\mathcal{N}$. For this, let us assume, by contradiction, that the sign of f remains the same when we evaluate it either in the points of $\mathcal{M}$ or in the points of $\mathcal{N}$. Without loss of generality, we may assume that

$$\text{if} \quad \vec{v} \in \mathcal{M} \cup \mathcal{N}, \quad \text{then} \quad f(\vec{v}) > 0.$$

Define the auxiliary real valued composition function

$$g : (-\delta; \delta) \to \mathcal{M} \cup \{\vec{p}_1\} \cup \mathcal{N} \to \mathbb{R}, \quad g(t) := f(\vec{p}_1 + t\vec{u}),$$

for which it holds that its derivative at the point 0 is given by the directional derivative in the direction of $\vec{u}$,

$$\frac{dg}{dt}(0) = \nabla f(\vec{p}_1) \cdot \overrightarrow{u} = \nabla f(\vec{p}_1) \cdot \frac{\nabla f(\vec{p}_1)}{||\nabla f(\vec{p}_1)||} = ||\nabla f(\vec{p}_1)|| \, .$$

On the other hand, $g(t) > 0 = g(0)$, for all $t \in (-\delta; \delta) - \{0\}$, and then g is a real differentiable function which has a relative minimum at 0. Hence,

$$0 = \frac{dg}{dt}(0) = ||\nabla f(\vec{p}_1)|| \, ,$$

and so $\nabla f(\vec{p}_1) = \vec{0}$ which is a contradiction since $\vec{p}_1$ is regular. Whence, f takes different signs depending on whether it evaluates in $\mathcal{M}$ or in $\mathcal{N}$. Again, without loss of generality, we may assume that, if $\vec{v} \in \mathcal{M}$, then $f(\vec{v}) > 0$, and if $\vec{v} \in \mathcal{N}$, then $f(\vec{v}) < 0$. Therefore, $\mathcal{M} \subset B$ and $\mathcal{N} \subset C$. So, $B \neq \emptyset$ and $C \neq \emptyset$.

To continue with the proof of the first statement, let us prove now that $\vec{p}_1$ is a boundary point of B. For this, let us begin giving an $\varepsilon > 0$. Since, if $\vec{p} \in B$, it holds that $f(\vec{p}) > 0$, and $f(\vec{p}_1) = 0$, then $\vec{p}_1 \in B^c$. Hence, $\vec{p}_1 \in B(\vec{p}_1; \varepsilon) \cap B^c$, and by this $B(\vec{p}_1; \varepsilon) \cap B^c \neq \emptyset$. Now, let $\eta = \min(\delta, \varepsilon)$. Let us define

$$\mathcal{P} = \{\vec{v} \in \mathbb{R}^n : \text{there exists} \quad t \in (0; \eta) \quad \text{such that} \quad \vec{v} = \vec{p}_1 + t\vec{u}\}.$$

It is clear that $\mathcal{P} \neq \emptyset$, besides that $\mathcal{P} \subset \mathcal{M}$; and by this $\mathcal{P} \subset B$. On the other hand, if $\vec{v} \in \mathcal{P}$, it happens that

$$\|\vec{v} - \vec{p}_1\| = \|\vec{p}_1 + t\vec{u} - \vec{p}_1\| = \|t\vec{u}\| = |t| = t < \eta \leq \varepsilon.$$

In brief, $\|\vec{v} - \vec{p}_1\| < \varepsilon$; that is, $\vec{v} \in B(\vec{p}_1; \varepsilon)$. Since this is valid for *every* $\vec{v} \in \mathcal{P}$, we conclude that $\mathcal{P} \subset B(\vec{p}_1; \varepsilon)$ obtaining that

$$\emptyset \neq \mathcal{P} \subset B(\vec{p}_1; \varepsilon) \cap B.$$

Therefore, for every $\varepsilon > 0$ it holds that

$$B(\vec{p}_1; \varepsilon) \cap B \neq \emptyset \quad \text{and} \quad B(\vec{p}_1; \varepsilon) \cap B^c \neq \emptyset.$$

Hence, $\vec{p}_1$ is a boundary point of B; that is, $\vec{p}_1 \in \partial B$. Analogously, one may prove for C that $\vec{p}_1 \in \partial C$.

We now prove 2. Let us assume that $\vec{p}_1$ is an isolated critical point, but that it is not doubly isolated critical point of A. In this occasion, there exists $\delta > 0$ such that, in $B(\vec{p}_1; \delta)$, it happens that $\vec{p}_1$ is the only critical point of f. Starting with the subject,

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likewise the previous proof of statement (1), let us begin by proving that B and C are not empty (notice that in contrast to the previous case $\nabla f(\vec{p}_1) = 0$). For this let us assume, by contradiction, that $B = \emptyset$ or $C = \emptyset$. Without loss of generality we may assume that $B = \emptyset$. Hence by Lemma 1 $A \cup C \cup B = \mathbb{R}^n$. Since $B(\vec{p}_1; \delta) \cap B = \emptyset$, it holds that $B(\vec{p}_1; \delta) \subset A \cup C$. Let us prove that being B empty implies that

$$(B(\vec{p}_1; \delta) - \{\vec{p}_1\}) \cap A = \emptyset. \tag{11}$$

To begin with, we use again a proof by contradiction. If we assume that

$$(B(\vec{p}_1; \delta) - \{\vec{p}_1\}) \cap A \neq \emptyset,$$

then there exists $\vec{q} \in (B(\vec{p}_1; \delta) - \{\vec{p}_1\}) \cap A$. Moreover, $\vec{q} \in B(\vec{p}_1; \delta)$, $\vec{q} \neq \vec{p}_1$, which implies that $\vec{q}$ cannot be a singular point of f, that is, $\vec{q}$ is a regular point of f. Now, since $\vec{q} \in A$, it follows from the proved statement (1), that $\vec{q}$ is a boundary point of B. That is, $q \in \partial B$ and $\partial B \neq \emptyset$ which contradicts that $\partial B = \emptyset$ since $B = \emptyset$. Hence, (11) is proved; that is, $\vec{p}_1$ is an isolated point of A; which means that $\vec{p}_1$ is a doubly isolated critical point of A contradicting our hypothesis. Hence, $B \neq \emptyset$. Similarly, one may prove that $C \neq \emptyset$.

Let us proceed to the demonstration that $\vec{p}_1$ is a boundary point of B and C; which again it suffices to provide only for the case of B. Let us give $\varepsilon > 0$; and let us define $\eta = \text{Min}(\delta, \varepsilon)$. Since, in this case, $\vec{p}_1$ is not a doubly isolated critical point of A, it cannot be an isolated point of A, in particular we have that $(B(\vec{p}_1; \eta) - \{\vec{p}_1\}) \cap A \neq \emptyset$. As we have examine before, this implies that there exists a regular point $\vec{q}$ of f such that $\vec{q} \in B(\vec{p}_1; \eta) \cap A$; more precisely, $\vec{q} \in B(\vec{p}_1; \eta)$ and it is a regular point in A. By the first fact, there exists $\gamma > 0$ small enough such that $B(\vec{q}; \gamma) \subset B(\vec{p}_1; \eta)$. By statement (1), we have that $\vec{q}$ is a boundary point of B, in particular, $B(\vec{q}; \gamma) \cap B \neq \emptyset$ and $B(\vec{q}; \gamma) \cap B^c \neq \emptyset$. But then

$$\emptyset \neq B(\vec{q}; \gamma) \cap B \subset B(\vec{p}_1; \eta) \cap B \subset B(\vec{p}_1; \varepsilon) \cap B$$

and

$$\emptyset \neq B(\vec{q}; \gamma) \cap B^c \subset B(\vec{p}_1; \eta) \cap B^c \subset B(\vec{p}_1; \varepsilon) \cap B^c.$$

In brief, $B(\vec{p}_1; \varepsilon) \cap B \neq \emptyset$ and $B(\vec{p}_1; \varepsilon) \cap B^c \neq \emptyset$. Since this happens for all $\varepsilon > 0$, we conclude that $\vec{p}_1$ is boundary point of B. As usual, we omit the analogous proof for C; by virtue of which $\vec{p}_1 \in \partial C$.

Proof of 3. Let us assume that $n \geq 2$ and that $\vec{p}_1$ is an isolated point of A. In this case, by the second fact, there exists $\delta > 0$ such that, in $B(\vec{p}_1; \delta)$, it holds that $\vec{p}_1$ is the only point of A. Now, from Lemma 1 it follows, on one hand, that $A^c = B \cup C$ and hence $B(\vec{p}_1; \delta) - \{\vec{p}_1\} \subset B \cup C$; and, on the other hand, B and C are separate, and since $B(\vec{p}_1; \delta) - \{\vec{p}_1\}$ is connected in dimension greater or equal than two, it must happen two cases

$$B(\vec{p}_1; \delta) - \{\vec{p}_1\} \subset B \qquad \text{or} \qquad B(\vec{p}_1; \delta) - \{\vec{p}_1\} \subset C. \tag{12}$$

Without loss of generality, let us assume that $B(\vec{p}_1; \delta) - \{\vec{p}_1\} \subset B$. Whence, $B(\vec{p}_1; \delta) \cap C = \emptyset$; which implies that $\vec{p}_1$ cannot be already a boundary point of C. Let us prove that, in contrast, $\vec{p}_1$ is a boundary point of B. Let us give $\varepsilon > 0$ and let us define $\gamma = \min(\delta, \varepsilon)$. Since $\vec{p}_1 \in B(\vec{p}_1; \gamma)$ and $\vec{p}_1 \in A \subset B^c$, we have that $\vec{p}_1 \in B(\vec{p}_1; \gamma) \cap B^c$; in other words, $B(\vec{p}_1; \gamma) \cap B^c \neq \emptyset$. So,

$$\emptyset \neq B(\vec{p}_1; \gamma) \cap B^c \subset B(\vec{p}_1; \epsilon) \cap B^c.$$

Hence, $B(\vec{p}_1;\varepsilon)\cap B^c \neq \emptyset$. On the other hand, since $B(\vec{p}_1;\gamma)-\{\vec{p}_1\}\subset B(\vec{p}_1;\delta)-\{\vec{p}_1\}\subset B$, it happens that $B(\vec{p}_1;\gamma)\cap B\neq\emptyset$. By this

$$\emptyset \neq B(\vec{p}_1;\gamma)\cap B \subset B(\vec{p}_1;\epsilon)\cap B.$$

That is, $B(\vec{p}_1;\varepsilon)\cap B\neq\emptyset$. In brief, $B(\vec{p}_1;\varepsilon)\cap B\neq\emptyset$ and $B(\vec{p}_1;\varepsilon)\cap B^c\neq\emptyset$. Since this holds for all $\varepsilon>0$, we conclude that $\vec{p}_1$ is a boundary point of B. In conclusion, $\vec{p}_1$ is a boundary point of B but not of C. Analogously, one may prove the case in (12) for which $B(\vec{p}_1;\delta)-\{\vec{p}_1\}\subset C$, that $\vec{p}_1$ is a boundary point of C but not of B. Therefore, $\vec{p}_1$ only may be a boundary point exclusively of one of the subsets B or C.

Finally, notice that the following equivalent statements prove the last biconditional only for B:

a) $\vec{p}_1$ is a boundary point exclusively of B.

b) There exists $\delta>0$ such that $B(\vec{p}_1;\delta)-\{\vec{p}_1\}\subset B$.

c) There exists $\delta>0$ such that, if $\vec{p}\in B(\vec{p}_1;\delta)-\{\vec{p}_1\}$, then $f(\vec{p})>f(\vec{p}_1)$.

d) $f(\vec{p}_1)$ is a local minimum.

The equivalence $a)\Leftrightarrow b)$ follows from the above proof. The equivalences $b)\Leftrightarrow c)$ and $c)\Leftrightarrow d)$ are given by definition of B.

Finally, since $f(\vec{p}_1)$ is a local maximum or a local minimum, we have that $\nabla f(\vec{p}_1)=\vec{0}$ and that, if $\vec{p}\in B(\vec{p}_1;\delta)-\{\vec{p}_1\}$, then $\nabla f(\vec{p})\neq\vec{0}$. In other words, $\vec{p}_1$ is an isolated critical point of f; whence $\vec{p}_1$ is a doubly isolated critical point of A. This completes the proof.

3.2.2 The no doubly isolation in level sets as enough condition to be boundaries

Theorem 1 has topological implications which are stated in the following corollaries.

Corollary 1. Assume $n\geq 2$ and that the only critical points of f which are in A are isolated, we have the following facts:

1. $\mathrm{Iso}(A)=\partial B\,\triangle\,\partial C$, where $\triangle$ denotes the symmetric difference.
2. $A'=\partial B\cap\partial C$.
3. $A=\partial B\cup\partial C$.

Proof. We prove 2. Notice that from Lemma 1 (5) we have $\partial B\cup\partial C\subset A$. Since A is closed, $A=A'\cup Iso(A)$, which is a disjoint union. From (3) in Theorem 1, $Iso(A)\cap\partial B\cap\partial C=\emptyset$, implying that $\partial B\cap\partial C\subset A'$. By definition, the points in A' satisfy the condition to be in $\partial B\cap\partial C$.

We prove 3. From (3) in Theorem 1, $Iso(A)=(Iso(A)\cap\partial B)\cup(Iso(A)\cap\partial C)$ and $Iso(A)\cap\partial B\cap\partial C=\emptyset$, that is, we have $(Iso(A)\cap\partial B)\cap(Iso(A)\cap\partial C)=$

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$\emptyset$. Therefore, $Iso(A) \cap \partial B$ and $Iso(A) \cap \partial C$ are disjoint, which implies that their symmetric difference is $Iso(A)$, that is

$$\begin{aligned} Iso(A) &= (Iso(A) \cap \partial B) \triangle (Iso(A) \cap \partial C) \\ &= (Iso(A) \cap \partial B) \cup (Iso(A) \cap \partial C) \\ &= Iso(A) \cap (\partial B \cup \partial C). \end{aligned}$$

Hence, $Iso(A) \subset \partial B \cup \partial C$. Noting that A is closed, and using 2, and the fact that $\overline{A} = A' \cup Iso(A)$ we obtain

$$\begin{aligned} A &= A' \cup Iso(A) \\ &\subset (\partial B \cap \partial C) \cup (\partial B \cup \partial C) \\ &= \partial B \cup \partial C, \end{aligned}$$

and $\partial B \cup \partial C \subset A$ from Lemma 1 (5). This proves 3.

Finally, we prove 1. Substituting 2 and 3 in $Iso(A) = A \setminus A'$, it follows that

$$\begin{aligned} Iso(A) &= (\partial B \cup \partial C) \setminus (\partial B \cap \partial C) \\ &= \partial B \triangle \partial C \end{aligned}$$

which proves 1. ■

Corollary 2. If the only critical points of f which are in A are isolated but not doubly isolated, then $B \neq \emptyset$, $C \neq \emptyset$ and $\partial B = \partial C = A$.

Proof. By (2) of Theorem 1, we have that the points of A are simultaneously boundary points of B and C; that is, $A \subset \partial B$ and $A \subset \partial C$. Besides, it follows from Lemma 1 that $\partial B \subset A$ and $\partial C \subset A$. Hence $\partial B = \partial C = A$. Now, since $A \neq \emptyset$, the previous facts imply that $B \neq \emptyset$ and $C \neq \emptyset$. ■

3.3 The topological closure and regions defined by non strict inequalities

Let $n \geq 2$, $k \in \mathbb{R}$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable map, with $A \neq \emptyset$, $D = R_{\geq k}$ and $E = R_{\leq k}$.

Corollary 2 gives enough conditions on the critical points of f which are in A in order to permit A to be the common boundary of B and C. Hence, one notes that the topological closure translates into considering the non strict inequalities instead of the strict inequalities which define B and C, more precisely,

$$\begin{aligned} \overline{f^{-1}((k;+\infty))} &= \overline{B} \\ &= B \cup \partial B \\ &= B \cup A \\ &= D \\ &= f^{-1}([k;+\infty)) \end{aligned}$$

and similarly $\overline{f^{-1}((-\infty;k))} = \overline{C} = E = f^{-1}((-\infty;k]))$.

Now, emerges the question if one may recover the sets B and C by taking the corresponding interiors of D and E, under the same hypotheses of Corollary 2. The answer is affirmative (see Corollary 3); however, we rather prefer to get a more general result, which includes this particular case, by simplifying the assumptions of this corollary by taking $n \geq 2$. That is what we do in the Theorem 2 as explained in the introduction, whose proof is given below.

3.3.1 Proof of main Theorem 2

We prove only the case for D, that is we prove (5). Similar reasoning proves (6).

Interior: In order to establish the equality $D^\circ = B \cup (\mathrm{Iso}\,(A) \cap \partial B)$, we have to prove the inclusions $D^\circ \subset B \cup (\mathrm{Iso}\,(A) \cap \partial B)$ and $B \cup (\mathrm{Iso}\,(A) \cap \partial B) \subset D^\circ$.

($\subset$) Let $\vec{p} \in D^\circ$. Since $D^\circ \subset D = B \cup A$, $\vec{p} \in B$ or $\vec{p} \in A$. Assume that $\vec{p} \in A$. Since $\vec{p} \in D^\circ$, there exists $\delta > 0$ such that $B\,(\vec{p};\delta) \subset D$; but $D = C^c$ by (2) of Lemma 1, and hence $\vec{p}$ is not a boundary point of C. In summary, $\vec{p}$ is a point of A which is not a boundary point of C. Because $n \geq 2$, by (1) of Corollary 1, $\vec{p}$ must be an isolated point of A which is a boundary point exclusively of B; that is $\vec{p} \in \mathrm{Iso}\,(A) \cap \partial B$. In conclusion, we obtain $\vec{p} \in B \cup (\mathrm{Iso}\,(A) \cap \partial B)$. Since this happens for *every* $\vec{p} \in D^\circ$, we conclude that $D^\circ \subset B \cup (\mathrm{Iso}\,(A) \cap \partial B)$.

($\supset$) Let $\vec{p} \in B \cup (\mathrm{Iso}\,(A) \cap \partial B)$; that is $\vec{p} \in B$ or $\vec{p} \in \mathrm{Iso}\,(A) \cap \partial B$. If $\vec{p} \in B$, since $B \subset D$ and B is open, it occurs that $B = B^\circ \subset D^\circ$. Hence $\vec{p} \in D^\circ$. Now, if $\vec{p} \in \mathrm{Iso}\,(A) \cap \partial B$, this means that $\vec{p} \in A$ and that there exists $\delta > 0$ such that

$$(B\,(\vec{p};\delta) - \{\vec{p}\}) \cap A = \emptyset \qquad \text{and} \qquad (B\,(\vec{p};\delta) - \{\vec{p}\}) \cap B \neq \emptyset.$$

Hence, since $\vec{p} \notin B$, there is $\delta_0 > 0$ such that $B\,(\vec{p};\delta_0) - \{\vec{p}\} \subset B \subset D^\circ$ and

$$B\,(\vec{p};\delta_0) = \{\vec{p}\} \cup (B\,(\vec{p};\delta_0) - \{\vec{p}\}) \subset A \cup D^\circ \subset A \cup D = D.$$

In other words, $\vec{p} \in D^\circ$. In conclusion, in any case we obtain that $\vec{p} \in D^\circ$. Since this happens for *every* $\vec{p} \in B \cup (\mathrm{Iso}\,(A) \cap \partial B)$, we conclude that $B \cup (\mathrm{Iso}\,(A) \cap \partial B) \subset D^\circ$.

From the two inclusions $D^\circ \subset B \cup (\mathrm{Iso}\,(A) \cap \partial B)$ and $B \cup (\mathrm{Iso}\,(A) \cap \partial B) \subset D^\circ$, we get the equality $D^\circ = B \cup (\mathrm{Iso}\,(A) \cap \partial B)$. Besides, we may substitute $\mathrm{Iso}\,(A) \cap \partial B$ by $\mathrm{Iso}\,(A) - \partial C$ applying the equality between them, which is a consequence of (1) in Corollary 1.

Boundary: Since D is closed, we have that $\partial D = D - D^\circ$. Moreover, from the above proof, $D^\circ = B \cup (\mathrm{Iso}\,(A) \cap \partial B)$. Hence by elementary set properties one has the following equalities

$$\begin{aligned} \partial D &= D - (B \cup (\mathrm{Iso}\,(A) \cap \partial B)) \\ &= (D - B) - (\mathrm{Iso}\,(A) \cap \partial B)\,. \end{aligned}$$

Finally, from the last equality and (2) in Lemma 1 it follows that $A = D - B$ and therefore $\partial D = A - (\mathrm{Iso}\,(A) \cap \partial B)\,.$

Exterior: Since D is closed, we have that D^c is open. Again, from (2) in Lemma 1 we have $(A \cup B)^c = C$ and then

$$\mathrm{Ext}\,(D) = (D^c)^\circ = D^c = (A \cup B)^c = C.$$

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This completes the proof of (5).

Corollary 3. Under the assumptions of Corollary 2 it satisfies that $\overline{B}^{\circ} = B$ and $\overline{C}^{\circ} = C$.

Proof. It follows from Theorem 2. ■

4. Conclusion

In this paper was proved that under certain conditions on the critical points for a differentiable function (not necessarily of class C^1) a corresponding level set is the whole boundary of its complementary associated open sets. The authors believe that there is at least one direction to continue with this research by considering complex valued functions in several complex variables, $f : \mathbb{C}^n \longrightarrow \mathbb{C}$, which are real differentiable but non analytic in the complex sense, in whose case the set of doubly isolated points must be empty.

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References

[1] Apostol T. M., Calculus, Vol. 2: Multi-Variable Calculus and Linear Algebra with Applications to Differential Equations and Probability, 2nd Ed, John Wiley & Sons, 1969.

[2] Baranenkov G. et al., Problems in Mathematical Analysis, Edited by B. Demidovich, Translated by G. Yankovsky, Gordon & Breach, 1970.

[3] Hirsch M. W., Differential Topology, Springer-Verlag, New York, 1997.

[4] Munkres J. R., Topology, 2nd Ed, Prentice-Hall, Inc. 2000.

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